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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

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On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 20 parts. The 20 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A solution of the Dirac equation for a free spinor field is of the form

$$\psi(x) = u(\vec{p}, s) e^{-ipx} \Big|_{p^0=\omega_p} .$$

In this problem the momentum p is always on-shell, $p^0 = \omega_p = \sqrt{\vec{p}^{\,2} + m^2}$.

1.1 Show that $\psi(x)$ is a solution of the Klein-Gordon equation.

1.2 The Dirac equation implies an equation for $u(\vec{p}, s)$. Determine this equation.

1.3 Show that the matrix γ^0 has two eigenvalues $+1$ and two eigenvalues -1 (do not use an explicit representation for the γ-matrices!).

1.4 The solution for $u(p, s)$ is

$$u(\vec{p}, s) = \frac{\gamma^\mu p_\mu + m}{\sqrt{2m(m + \omega_p)}} u(0, s) \,, \qquad (1.1)$$

with $s = 1, 2$, and

$$\gamma^0 u(0, s) = u(0, s) \,. \qquad (1.2)$$

Show that eq. (1.2) follows from eq. (1.1) in the restframe.

1.5 Choose the two independent solutions for $u(0,s)$ such that $u^\dagger(0,s)u(0,t) = \delta_{st}$, $s,t = 1,2$. Show that

$$\bar{u}(\vec{p},s)u(\vec{p},t) = \delta_{st}\,.$$

PROBLEM 2

Consider the following Lagrangian density for a quantum field theory involving two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + a\partial_\mu\phi_1\partial^\mu\phi_2 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - 2m^2\phi_1\phi_2.$$

where a is a constant.

2.1 Determine the equations of motion for ϕ_1 and ϕ_2.

2.2 What are the canonical momenta π_1 and π_2 associated to ϕ_1 and ϕ_2?

2.3 Express the Hamiltonian in terms of the canonical coordinates and momenta.

2.4 Give the result of 1.3 for the special case $a = 0$.

2.5 Show that for $a = 1$ the equations of motion imply that $\phi_1 = \phi_2$. What is in this case the mass of the field $\phi \equiv \phi_1 + \phi_2$?

PROBLEM 3

The Lagrangian density for the Dirac field is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\,.$$

3.1 Define the canonical momentum corresponding to the field ψ, and show that it equals

$$\pi(t,\vec{x}) = i\psi^\dagger(t,\vec{x})\,.$$

3.2 The Hamiltonian is defined as

$$H = \int d^3x\,(\pi(t,\vec{x})\partial_0\psi(t,\vec{x}) - \mathcal{L})\,.$$

Show that this equals

$$H = -\int d^3x\,\bar{\psi}(i\gamma^k\partial_k - m)\psi\,.$$

3.3 The invariance of $\mathcal{L}$ under transformations

$$\psi \rightarrow \psi' = e^{i\theta}\psi$$

gives rise to a current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$. Show that, if the Dirac equation for ψ (and $\bar{\psi}$) holds, j^μ satisfies

$$\partial_\mu j^\mu = 0\,.$$

3.4 Show that

$$Q = \int d^3x\, j^0$$

is constant in time if ψ and its spatial derivatives go sufficiently fast to zero at large $|\vec{x}|$.

3.5 Show that for four arbitrary operators A, B, C, D the following commutation relation holds:

$$[AB, CD] = -C\{D, A\}B + \{C, A\}DB - AC\{D, B\} + A\{C, B\}D\,.$$

3.6 The equal-time anticommutation relations for the Dirac field are

$$\{\psi_a(t,\vec{x}), \pi_b(t,\vec{y})\} = i\delta_{ab}\delta^3(\vec{x}-\vec{y})\,,\ \{\psi_a(t,\vec{x}), \psi_b(t,\vec{y})\} = \{\pi_a(t,\vec{x}), \pi_b(t,\vec{y})\} = 0\,.$$

Show that under the same conditions as in (3.4)

$$[H, Q] = 0\,.$$

PROBLEM 4

In scattering theories one descibes the ingoing (outgoing) particles by free fields ϕ_{in} (ϕ_{out}). To these fields correspond in (out) creation- and annihilation-operators, and Fock spaces of in- and out-states. The S-matrix is a map from the out- to the in-states:

$$|\alpha, \text{in}> = S|\alpha, \text{out}>\,,$$

where α denotes a set of properties (momenta,...) the states might have. The states $|\alpha, \text{in}>$ ($|\alpha, \text{out}>$) satisfy the usual orthogonality ($<\beta, \text{in}|\alpha, \text{in}> = \delta_{\alpha\beta}$) and completeness ($\sum_\alpha |\alpha, \text{in}><\alpha, \text{in}| = 1$) relations.

4.1 Show that $S^\dagger S = SS^\dagger = 1$.

4.2 Show that $\phi_{\text{in}} = S\phi_{\text{out}}S^\dagger$.

4.3 The interacting field $\phi(x)$, which interpolates between ϕ_{in} and ϕ_{out}, is related to ϕ_{in} by a unitary operator $U(t)$, with $U(t=-\infty)=1$:

$$\phi(x) = U^\dagger(t)\phi_{\text{in}}(x)U(t)\,.$$

What is the relation between $\phi(x)$ and ϕ_{out}?

4.4 What is the limit of $U(t)$ for $t \to +\infty$?